Gauthiers.

7.26. Compute $\sum_{\text{all spins}} [\bar{v}(a)\Gamma_1 v(b)][\bar{v}(a)\Gamma_2 v(b)]^*$,

$\sum_{\text{all spins}} [\bar{u}(a)\Gamma_1 v(b)][\bar{u}(a)\Gamma_2 v(b)]^*$,

$\sum_{\text{all spins}} [\bar{v}(a)\Gamma_1 u(b)][\bar{v}(a)\Gamma_2 u(b)]^*$

with Casimir's trick.

$[\bar{v}(a)\Gamma_1 v(b)][\bar{v}(a)\Gamma_2 v(b)]^*$

$= \bar{v}(a)\Gamma_1 v(b)\, v^*(b)\,\gamma^0\,\Gamma_2^*\,\gamma^0\, v(a)$

$= \bar{v}(a)\Gamma_1 v(b)\,\bar{v}(b)\,\bar{\Gamma}_2 v(a)$.

Sum over all spins $\Rightarrow \sum_a \bar{v}(a)\Gamma_1 \left\{ \sum_b v(b)\bar{v}(b) \right\} \bar{\Gamma}_2 v(a)$

$= \sum_a \bar{v}(a)\left[\Gamma_1 (\not{p}_b - mc)\,\bar{\Gamma}_2\right] v(a)$.

$= \sum_a \bar{v}(a)_i\, Q_{ji}\, v(a)_j$

$= \sum_a \bar{v}(a)_i\, v(a)_j\, Q_{ji}$

$= \sum_a \mathrm{Tr}\left[\bar{v}(a)_i v(a)_j\right] Q_{ji}$

$= \sum_a \mathrm{Tr}\left[\left[\bar{v}(a)v(a)\right]_{kj} Q_{ji}\right]$

$= \mathrm{Tr}\left[\left\{\sum_a \left(\bar{v}(a)v(a)\right)_{kj}\right\} Q_{ji}\right]$.

$$\mathrm{Tr}\left[\sum_a (\bar{v}(a) v(a))_{kj} \, Q_{ji} \right]$$

$$= \boxed{\mathrm{Tr}\left[(\not{p}_a - m_a c)_{kj} \left[\Gamma_1 (\not{p}_b - m_b c) \bar{\Gamma}_2 \right]_{ji} \right]}$$

More succinctly, $\mathrm{Tr}\left[(\not{p}_a - m_a c) \Gamma_1 (\not{p}_b - m_s c) \bar{\Gamma}_2 \right]$

$$\sum_{\text{all spins}} \left[\bar{u}(a) \Gamma_1 v(b) \right] \left[\bar{u}(a) \Gamma_2 v(b) \right]^*$$

$$= \sum \bar{u}_a \Gamma_1 v_b \, v_b^* \, \gamma^0 \Gamma_2^* \gamma^0 u(a)$$

$$= \sum \bar{u}_a \Gamma_1 v_b \bar{v}_b \bar{\Gamma}_2 u_a .$$

$$= \sum_a \bar{u}_a \left\{ \sum_b \Gamma_1 v_b \bar{v}_b \bar{\Gamma}_2 \right\} u_a .$$

$$= \sum_a \bar{u}_a \left[\Gamma_1 (\not{p}_b - m c) \bar{\Gamma}_2 \right] u_a$$

$$= \sum_a \bar{u}_{a\,i} \left[\Gamma_1 (\not{p}_b - m c) \bar{\Gamma}_2 \right]_{ij} u_{a\,j} .$$

$$= \sum_a \delta_{jk} \left(\bar{u}_a u_a \right)_{ji} \left[\Gamma_1 (\not{p}_b - m c) \bar{\Gamma}_2 \right]_{ik}$$

$$= \mathrm{Tr}\left\{ \left[\sum_a (\bar{u}_a u_a) \right]_{ji} \left[\Gamma_1 (\not{p}_b - m c) \bar{\Gamma}_2 \right]_{ik} \right\}$$

$$= \boxed{\mathrm{Tr}\left\{ (\not{p}_a + m c)_{ji} \left(\Gamma_1 (\not{p}_b - m c) \bar{\Gamma}_2 \right)_{ik} \right\}}$$

$$\sum_{\text{all spins}} [\bar{v}_a \Gamma_1 u_b][\bar{v}_a \Gamma_2 u_b]^*$$

$$= \sum \bar{v}_a \Gamma_1 u_b u_b^* \gamma^0 \gamma^0 \Gamma_2^* \gamma^0 v_a$$

$$= \sum_a \bar{v}_a \Gamma_1 \left\{ \sum_b u_b \bar{u}_b \right\} \bar{\Gamma}_2 v_a$$

$$= \sum_a \bar{v}_a \Gamma_1 (\not{p}_b + m_b c) \bar{\Gamma}_2 v_a$$

$$= \sum_a \bar{v}_{a\,i} \left[\Gamma_1 (\not{p}_b + m_b c) \bar{\Gamma}_2 \right]_{ij} v_{a\,j}$$

$$= \sum_a \left[\bar{v}_{a\,i} v_{a\,j} \right] \left[\Gamma_1 (\not{p}_b + m_b c) \bar{\Gamma}_2 \right]_{ij}$$

$$= \sum_a \left[\bar{v}_a v_a \right]_{ik} \delta_{kj} \left[\Gamma_1 (\not{p}_b + m_c) \bar{\Gamma}_2 \right]_{ij}$$

$$= \sum_a \delta_{kj} \left[\bar{v}_a v_a \right]_{ik} \left[\Gamma_1 (\not{p}_b + m_c) \bar{\Gamma}_2 \right]_{ij}$$

$$= \cancel{\sum_d} \Gamma_k \left\{ \sum_a \left[\bar{v}_a v_a \right]_{ik} \left[\Gamma_1 (\not{p}_b + m_c) \bar{\Gamma}_2 \right]_{ij} \right\}$$

$$= \boxed{\Gamma_k \left\{ (\not{p}_a - m_a c)(\Gamma_1 (\not{p}_b + m_b c) \bar{\Gamma}_2) \right\}}$$